Loo-Keng Hua Lectures

Lecture 1

Geometric local class field lectures

Classical geometric class field theory

X/k smooth projective curve

$\uparrow$ alg. closed

$$\boxed{\underline{\text{Th}}: \ \pi_1(X)^{ab} \simeq \pi_1(\mathrm{Jac}_X)}$$

$\underline{\text{Example}}$: $X/\mathbb{C}$ $\quad \mathrm{Jac}_X^{an} \simeq \underbrace{H^1(X, \mathcal{O}_X)}_{\text{Tangent space at } 0 \text{ of } \mathrm{Jac}_X} / \Lambda$

$$\begin{array}{ccccc} \Lambda = \pi_1(\mathrm{Jac}_X^{an}) & \subset & H^1(X, \mathcal{O}_X) \\ \| & & \| \text{ Serre duality} \\ \pi_1(X^{an})^{ab} = H_1(X^{an}, \mathbb{Z}) & \subset & H^0(X, \Omega^1_X)^* \end{array}$$

Geometric Langlands point of view

$\mathcal{E}/X$ rk. 1 $\overline{\mathbb{Q}}_\ell$-étale local system.

$d \geq 1$ $\quad \mathrm{Div}^d = X^d/\sigma_d$ = Hilbert scheme of degree d effective divisors /X

$$\mathrm{Div}^1 = X$$

$$\Sigma^d : X^d \twoheadrightarrow \mathrm{Div}^d$$ symmetrization morphism

$$(x_1, \dots, x_d) \longmapsto \sum_{i=1}^{d} [x_i]$$

$$\mathrm{Pic} = \coprod_{d \in \mathbb{Z}} \mathrm{Pic}^d$$ Coarse moduli space of line bundles/X

$$\mathrm{Pic}^0 = \mathrm{Jac}\, X$$

$$AJ^d : \mathrm{Div}^d \longrightarrow \mathrm{Pic}^d$$ Abel-Jacobi morphism

$$D \longmapsto \mathcal{O}(D)$$

$$\mathcal{E} \longmapsto \mathcal{E}^{(d)} := \Big(\underbrace{\textstyle\sum_*^d \mathcal{E}^{\boxtimes d}}_{\substack{\mathfrak{S}_d\text{- equivariant} \\ \text{loc. system on } X^d}}\Big)^{\mathfrak{S}_d}$$

sheaf with action of "abstract" on Div^d.

$\mathcal{E}^{(d)}$ = Nb. 1 $\overline{\mathbb{Q}}_\ell$-local on Div^d.

= perverse sheaf on Div^d in general when $\mathrm{nb}(\mathcal{E}) > 1$.

Rem: $*$ X/k alg. variety, if $\mathrm{char}(k) > 0$.
$d \geq 2$, $\pi_1(X^d/\mathfrak{S}_d) = \pi_1(X)^{ab}$ (SGA 1. Ch. IX Rem. 5.8)

$*$ X CW complex, $d \geq 2$
$\pi_1(X^d/\mathfrak{S}_d) = \pi_1(X)^{ab}$

$\Longrightarrow$ Symmetrization abelianizes the π_1.

For $d > 2g_X - 2$, AJ^d is a locally trivial fibration in $\underbrace{\mathbb{P}^{d-g_X}}_{\text{simply connected}}$

$\Rightarrow$ $\mathcal{E}^{(d)}$ descends to a rk. 1 $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{F}^{(d)}$ on Pic^d,

$$\boxed{\mathcal{E}^{(d)} = AJ^{d*}\mathcal{F}^{(d)}}$$

$$\mathcal{F}^{>2g-2} = \coprod_{d>2g-2} \mathcal{F}^{(d)} \text{ on } \mathrm{Pic}^{>2g-2} = \coprod_{d>2g-2} \mathrm{Pic}^d$$

is equivariant on the monoid $\mathrm{Pic}^{>2g-2}$

$\rightsquigarrow$ $m : \mathrm{Pic}^{>2g-2} \times \mathrm{Pic}^{>2g-2} \longrightarrow \mathrm{Pic}^{>2g-2}$

$$m^*\mathcal{F}^{>2g-2} \simeq \mathcal{F}^{>2g-2} \boxtimes \mathcal{F}^{>2g-2}$$

\+ Compatibility relations (Cocycle relation)

\+ this monoid generates the group Pic

$\Rightarrow$ $\mathcal{F}^{>2g-2}$ extends naturally to a rk. 1 equivariant $\overline{\mathbb{Q}}_\ell$ - local system / Pic

$$\mathcal{F} = \coprod_{d\in\mathbb{Z}} \underbrace{\mathcal{F}^{(d)}}_{\text{on } \mathrm{Pic}^d}$$

$\rightsquigarrow$ $\mathcal{F}^{(0)}$ = rk. 1 $\overline{\mathbb{Q}}_\ell$ - local system / Jac_X

<u>En résumé:</u>

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, rightsquigarrow, "\text{symmetric powers}"] \arrow[dd, dashed] & & \mathcal{E}^{(d)} \arrow[dd, rightsquigarrow, "\text{descent via } AJ^d"] \\
& & \\
\mathcal{F}^{(0)} & & \mathcal{F}^{(d)} \arrow[ll, leftsquigarrow, "\text{extension}", "\text{via group law on Pic}"']
\end{tikzcd}$$

$$\boxed{d >> 0}$$

Rem: Let $\mathcal{P}ic$ be the Picard stack

$$\mathcal{P}ic \longrightarrow \mathrm{Pic} \text{ is a } \mathbb{G}_m\text{-gerb}$$

Connected

(trivial after fixing a k-rat. point
$\mathcal{P}ic \simeq [\mathrm{Pic}/\mathbb{G}_m]$)

$\Rightarrow$ $\overline{\mathbb{Q}}_\ell$-local systems $/\mathrm{Pic} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$-local systems $/\mathcal{P}ic$.

In the next lectures, in my context this will be different: $\underbrace{\mathbb{Q}_\ell^\times}_{\text{totally disconnected}}$-gerb $\Rightarrow$ will have to work with the Picard stack not the coarse moduli space.

Example: $X/\mathbb{F}_q$ smooth proper geo. connected curve

The preceding geometric construction for $X_{\overline{\mathbb{F}}_q}$ is compatible with Frobenius and induces rk. 1 $\overline{\mathbb{Q}}_\ell$-Weil local systems / X

$\Big\downarrow$ arithmetic construction

equivariant rk. 1 $\overline{\mathbb{Q}}_\ell$-Weil local systems / Pic

$\wr$ trace of Frob function

Grothendieck's faisceau/functions dictionary

$\Big\downarrow$

Character $\underbrace{\mathrm{Pic}(\mathbb{F}_q)}_{F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_{F_v}^\times} \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ where $F = \mathbb{F}_q(X)$

$\mathcal{T}_1$ equivariant $\Rightarrow$ tr(σ_1) character

$\rightsquigarrow$ Defines dually the isomorphism of everywhere unramified class field theory

$$\boxed{F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_{F_v}^\times \xrightarrow{\sim} W_{F^{un}/F}^{ab}}$$

F^{un}/F max. unramified

$$\begin{tikzcd}
W_{F^{un}/F} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \pi_1(X) \arrow[d, two heads] \\
\mathrm{Frob}^{\mathbb{Z}} \arrow[r, hook] & \mathrm{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q)
\end{tikzcd}$$

To deduce the iso.: use $A/\mathbb{F}_q$ ab. var.

$$W_A^{ab} \simeq A(\mathbb{F}_q) \times \mathbb{Z}$$

[Lang

In fact $\pi_1(A_{\overline{\mathbb{F}}_q}) \simeq \prod_\ell T_\ell(A)$

$$\Rightarrow W_A^{ab} \simeq \underbrace{\mathrm{Coker}\left(\prod_\ell T_\ell(A) \xrightarrow{F-\mathrm{Id}} \prod T_\ell(A)\right)}_{A(\mathbb{F}_q)} \times \mathbb{Z}$$

Next: Prove local class field theory

$$\boxed{E^\times \xrightarrow{\sim} W_E^{ab}}$$ for $[E:\mathbb{Q}_p] < +\infty$
or $E = \mathbb{F}_q((\pi))$

by proving that for $d \gg 0$ Picard stacks of deg. d line bundles on the curve

$$\mathrm{Div}^d \xrightarrow{AJ^d} \mathrm{Pic}^d$$

is a [pro-étale locally trivial fibration in simply connected diamonds.]